بعض الخواص الهندسية لمؤثر تكاملي ميرومورفيك معروف

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الملخص:

في هـذا البحـث بعـض الخـواص الهندسـية لمـؤثر تكـاملي ميرومورفيـك معـروف داخـل القـرص المفتـوح درسـت، عـلاوة على ذلك بعـض الشـروط كي يكون هـذا المـؤثر التكاملي في فئـات من الـدوال الميرومورفيك قد حددت.

الكلمات المفتاحية: الدوال التحليلية، الدوال النجمية, الدوال المحدبة, قرص الوحدة المفتوح

Abstract:

In this paper, some geometric properties for certain of meromorphic integral Operator in the punctured open unit disk are studied. Moreover, some supplementary conditions for which the class of integral operators to be in subclasses of meromorphic functions are determined.

Keywords: Analytic function, starlike functions, convex functions, open unit disk.

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1- Introduction

Let U = {$z \in$ C : $|z| < 1$}, be the open unit disc in the complex plane C, $U^* = U \setminus \{0\}$, the punctured open unit disk and H(U) = {f : U −→ C : f is holomorphic in U}. For $a \in$ C and $n \in$ N (N = {0,1,2,...}), let H[a,n] = {$f \in$ H(U) ,

$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\ z \in U$. Let Σ denote the class of functions of the form

which are analytic in the punctured open unit disk U^*.

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \qquad (1.1)$$

We say that a function $f \in \Sigma$ is meromorphic starlike function of order $\alpha(0 \le \alpha < 1)$, and belongs to the class $\Sigma^*(\alpha)$, if it satisfies the inequality

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha. \qquad (1\text{-}2)$$

A function $f \in \Sigma$ is a meromorphic convex function of order $\alpha(0 \le \alpha < 1)$, if f satisfies the following inequality and we denote this class by $\Sigma_k(\alpha)$.

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad (1.3)$$

For $f \in \Sigma$, Wang et al. [11] introduced and studied the subclasses $\Sigma_M(\delta)$ and $\Sigma_N(\delta)$ of Σ as the following.

A function $f \in \Sigma$ is said to be in the class $\Sigma_M(\delta)$, $\delta > 1$, if f satisfies the following inequality:

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) < \delta \qquad (\delta > 1,\ z \in \mathbb{U}) \qquad (1.4)$$

A function $f \in \Sigma$ is said to be in the class $\Sigma_N(\delta)$, $\delta > 1$, if f satisfies the following inequality:

$$-\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) < \delta \qquad (\delta > 1,\ z \in \mathbb{U}) \qquad (1.5)$$

For $\alpha < 1 < \delta$, we define the following two subclasses of meromorphic functions ([3], [4] and [5]).

A function $f \in \Sigma$ is said to be in the class $\Sigma^*(\alpha, \delta)$, if f satisfies the following inequality:

$$\alpha < -\Re\left(\frac{zf'(z)}{f(z)}\right) < \delta,\ \ \alpha < 1 < \delta,\ \ z \in \mathbb{U} \tag{1.6}$$

A function $f \in \Sigma$ is said to be in the class $\Sigma K(\alpha,\delta)$, if f satisfies the following inequality:

$$\alpha < -\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) < \delta,\ \ \alpha < 1 < \delta,\ \ z \in \mathbb{U} \tag{1.7}$$

If we consider $\delta \to \infty$ in the subclasses $\sum^*(\alpha,\delta)$ and $\sum K(\alpha,\delta)$, we obtain respectively, the familiar subclasses $\sum^*(\alpha)$ and $\sum_k(\alpha)$, $0 \le \alpha < 1$. In addition, if we consider $\alpha \to 0$ in the subclasses $\sum^*(\alpha,\delta)$ and $\sum K(\alpha,\delta)$, we obtain respectively, the subclasses $\sum_M(\delta)$, and $\sum_N(\delta)$, $\delta > 1$.

In the literature several integral operators of meromorphic functions in the punctured open unit disk have been investigated and studied by many authors (cf., e.g., [1-2, 5-10]).

Mohammed and Darus, introduced and studied [1] the following meromorphic integral operator,

$$\mathcal{J}_{\beta,\gamma}(f)(z) = \left(\frac{2(\gamma-\beta)}{z^{2\gamma}}\int_0^z t^{2\gamma-1}(-f'(t))^{\beta}dt\right)^{\frac{1}{2\beta}} \tag{1.8}$$
$$\left(f \in \Sigma,\ \gamma,\beta \in \mathbb{C},\ \beta \in \mathbb{C}\backslash\{0\},\ \Re(\gamma-\beta) > \tfrac{1}{2}\right)$$

In this paper we will study the properties of the image of a function f belongs the above subclasses through the integral operator $J_{\beta,\gamma}$.

In order to prove our main results, we shall need the following lemmas due to Alina Totoi
[5]:

Lemma 1.1 *Let* $n \in \mathbb{N}^*, \alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{C}$ *with* $R\,[\gamma - \alpha\beta] \ge 0$. *If* $q \in H[q(0),n]$ *with* $q(0) \in \mathbb{R}$ *and* $q(0) > \alpha$, *then we have*

$$\Re\left\{q(z) + \frac{zq'(z)}{\gamma - \beta q(z)}\right\} > \alpha \Longrightarrow \Re q(z) > \alpha,\ \ z \in \mathbb{U} \tag{1.9}$$

Lemma 1.2 *Let* $n \in \mathbb{N}^*, \delta, \beta \in \mathbb{R}, \gamma \in \mathbb{C}$ *with* $<[\gamma-\delta\beta] \ge 0$. *If* $q \in H[q(0),n]$ *with* $q(0) \in \mathbb{R}$ *and* $q(0) < \delta$, *then we have*

$$\Re\left\{q(z) + \frac{zq'(z)}{\gamma - \beta q(z)}\right\} < \delta \Longrightarrow \Re q(z) < \delta,\ \ z \in \mathbb{U} \tag{1.10}$$

2- Main result

Our first result involving the properties of the image of a function $f \in \Sigma K(\alpha,\delta)$, through the integral operator $J_{\beta,\gamma}$ defined by (1.8).

Theorem 2.1. *Let* $\beta > 0, \gamma \in \mathbb{C}, \alpha < 1 < \delta$ *and* $\frac{1}{2}(\alpha+1) < 1 < \frac{1}{2}(\delta+1) \leq \frac{\Re\gamma}{\beta}$. *If* $f \in \Sigma K(\alpha,\delta)$, *then* $\mathcal{J}_{\beta,\gamma}(z) \in \Sigma^{\star}(\frac{1}{2}(\alpha+1), \frac{1}{2}(\delta+1))$, *where* $J_{\beta,\gamma}(z)$ *is the integral operator given by (1.8).*

Proof. From (1.8), we have

$$z^{2\gamma}\left(\mathcal{J}_{\beta,\gamma}(z)\right)^{2\beta} = 2(\gamma-\beta)\int_0^z t^{2\gamma-1}\left(-f'(t)\right)^{\beta}dt. \qquad (2.1)$$

Differentiating (2.1) with respect to z and by simple computation, we obtain

$$\left(\frac{z\mathcal{J}'_{\beta,\gamma}(z)}{\mathcal{J}_{\beta,\gamma}(z)} + \frac{\gamma}{\beta}\right)\left(\mathcal{J}_{\beta,\gamma}(z)\right)^{2\beta} = \frac{(\gamma-\beta)}{\beta}\left(-f'(z)\right)^{\beta} \qquad (2.2)$$

Differentiating (2.2) logarithmically and by simple computation, we get

$$\frac{\beta z^2\mathcal{J}''_{\beta,\gamma}(z) + (\beta+\gamma)z\mathcal{J}'_{\beta,\gamma}(z)}{\beta z\mathcal{J}'_{\beta,\gamma}(z)) + \gamma\mathcal{J}_{\beta,\gamma}(z)} + (2\beta-1)\frac{z\mathcal{J}'_{\beta,\gamma}(z))}{\mathcal{J}_{\beta,\gamma}(z)} = \beta\frac{zf''(z)}{f'(z)} \qquad (2.3)$$

We can write the left-hand side of (2.3), as the following:

$$\frac{\beta\left(\frac{z\mathcal{J}''_{\beta,\gamma}(z)}{\mathcal{J}'_{\beta,\gamma}(z)} + 1\right) + \gamma}{\beta + \gamma\frac{\mathcal{J}_{\beta,\gamma}(z)}{z\mathcal{J}'_{\beta,\gamma}(z)}} + (2\beta-1)\frac{z\mathcal{J}'_{\beta,\gamma}(z)}{\mathcal{J}_{\beta,\gamma}(z)} = \beta\frac{zf''(z)}{f'(z)} \qquad (2.4)$$

Let

$$q(z) = -\frac{z\mathcal{J}'_{\beta,\gamma}(z)}{\mathcal{J}_{\beta,\gamma}(z)} \qquad (2.5)$$

such that $q(z)$ is analytic in U, with $q(0) = 1$. Differentiating $q(z)$ logarithmically, we obtain

$$-q(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{z\mathcal{J}''_{\beta,\gamma}(z)}{\mathcal{J}'_{\beta,\gamma}(z)} \qquad (2.6)$$

Then (2.4) can be written as

$$q(z) + \frac{zq'(z)}{2\gamma - 2\beta q(z)} = \frac{1}{2}\left\{-\left(\frac{zf''(z)}{f'(z)} + 1\right)\right\} + \frac{1}{2} \qquad (2.7)$$

Since $f \in \Sigma K(\alpha,\delta)$, $\alpha < 1 < \delta$, then

$$\alpha < -\Re\left(\frac{zf''(z)}{f'(z)}+1\right) < \delta,\ z\in\mathbb{U} \tag{2.8}$$

Which is equivalent to

$$\frac{1}{2}(\alpha+1) < \frac{1}{2}\Re\left\{-\left(\frac{zf''(z)}{f'(z)}+1\right)\right\}+\frac{1}{2} < \frac{1}{2}(\delta+1) \tag{2.9}$$

Using (2.7) and (2.9) we get

$$\frac{1}{2}(\alpha+1) < \Re\left\{q(z)+\frac{zq'(z)}{2\gamma-2\beta q(z)}\right\} < \frac{1}{2}(\delta+1) \tag{2.10}$$

Since $\frac{1}{2}(\alpha+1) < q(0) = 1 < \frac{1}{2}(\delta + 1)$ and $0 \le 2\gamma - \beta(\delta + 1) < 2\gamma - \beta(\alpha + 1)$
We obtain from (2.10), after applying Lemma 1.1 and Lemma 1.2, that

$$\frac{1}{2}(\alpha+1) < \Re q(z) < \frac{1}{2}(\delta+1) \tag{2.11}$$

which is equivalent to

$$\frac{1}{2}(\alpha+1) < \Re\left\{-\frac{z\mathcal{J}_{\beta,\gamma}'(z)}{\mathcal{J}_{\beta,\gamma}(z)}\right\} < \frac{1}{2}(\delta+1) \tag{2.12}$$

That is $\mathcal{J}_{\beta,\gamma}(z) \in \Sigma^\star(\frac{1}{2}(\alpha+1), \frac{1}{2}(\delta+1))$.

If we consider $\alpha \to 0$ in the above theorem, we obtain the next corollary:

Corollary 2.2. *Let* $\beta > 0, \gamma \in \mathrm{C}, \delta > 1$ *and* $\frac{\Re\gamma}{\beta} \ge \frac{1}{2}(\delta+1) > 1$. *If* $f \in \Sigma_N(\delta)$, *then* $\mathcal{J}_{\beta,\gamma}(z) \in \Sigma_M\left(\frac{1}{2}(\delta+1)\right)$, *where* $J_{\beta,\gamma}(z)$ *is the integral operator given by (1.8).*

Theorem 2.3. *Let* $\beta < 0, \gamma \in \mathrm{C}, \alpha < 1 < \delta$, *and* $\frac{\Re\gamma}{\beta} \le \frac{1}{2}(\alpha+1) < 1 < \frac{1}{2}(\delta+1)$
If $f \in \Sigma K(\alpha,\delta)$, *then* $\mathcal{J}_{\beta,\gamma}(z) \in \Sigma^\star(\frac{1}{2}(\alpha+1), \frac{1}{2}(\delta+1))$, *where* $J_{\beta,\gamma}(z)$ *is the integral operator given by (1.8).*

Proof. From (2.7), we know that

$$q(z)+\frac{zq'(z)}{2\gamma-2\beta q(z)} = \frac{1}{2}\left\{-\left(\frac{zf''(z)}{f'(z)}+1\right)\right\}+\frac{1}{2} \tag{2.13}$$

Where $q(z) = -\frac{z\mathcal{J}_{\beta,\gamma}'(z)}{\mathcal{J}_{\beta,\gamma}(z)}$ is analytic in U.
By using the same steps as in the proof of Theorem 2.1 and since $f \in \Sigma K(\alpha,\delta), \alpha < 1 < \delta$, we receive

$$\frac{1}{2}(\alpha+1) < \Re\left\{q(z)+\frac{zq'(z)}{2\gamma-2\beta q(z)}\right\} < \frac{1}{2}(\delta+1) \tag{2.14}$$

Since $\frac{1}{2}(\alpha+1) < q(0) = 1 < \frac{1}{2}(\delta + 1)$ and $0 \leq 2\gamma - \beta(\delta + 1) < 2\gamma - \beta(\alpha + 1)$, then by applying Lemma 1.1 and Lemma 1.2, we obtain from (2.14) that

$\frac{1}{2}(\alpha+1) < \Re q(z) < \frac{1}{2}(\delta+1)$, where

$$q(z) = -\frac{z\mathcal{J}'_{\beta,\gamma}(z)}{\mathcal{J}_{\beta,\gamma}(z)}$$

That is $\mathcal{J}_{\beta,\gamma}(z) \in \Sigma^\star(\frac{1}{2}(\alpha+1), \frac{1}{2}(\delta+1))$

If we consider $\delta \longrightarrow \infty$ in the above theorem, we obtain the next corollary:

Corollary 2.4. *Let* $\beta < 0$, $\gamma \in$ C *and* $\frac{\Re\gamma}{\beta} \leq \frac{1}{2}(\alpha+1) < 1$. *If* $f \in \Sigma_k(\alpha)$, *then* $J_{\beta,\gamma}(z) \in \Sigma^\star(\frac{1}{2}(\alpha+1)$, *where* $J_{\beta,\gamma}(z)$ *is the integral operator given by (1.8).*

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